



## INTERNATIONAL JOURNAL OF RESEARCH IN COMMERCE, IT AND MANAGEMENT

### CONTENTS

Sr. No.	TITLE & NAME OF THE AUTHOR (S)	Page No.
1.	INFORMATION TECHNOLOGY AND ITS APPLICATION AMONG USERS & NON-USERS IN IRAN <i>DR. ALI BARATI DEVIN</i>	1
2.	ACADEMIC STAFF'S PERCEPTION OF ADMINISTRATIVE STAFF SERVICES IN ETHIOPIA: A CASE STUDY OF ADI-HAQI CAMPUS, MEKELLE UNIVERSITY <i>DR. TESFATSION SAHLU DESTA</i>	5
3.	XBRl, THE 21ST CENTURY DATA SOURCE AND DATABASE LEVEL DATA VALIDATION <i>FABOYEDE, S.O., MUKORO, D. &amp; OLOWE, O.</i>	15
4.	ORGANISATIONAL CULTURE MANACLES TO EMBARK UPON DURING GLOBAL CONDENSE <i>DR. A. CHANDRA MOHAN, DR. K. VASANTHI KUMARI &amp; DR. P. DEVARAJ</i>	22
5.	IMPACT OF REFORMS ON THE SOUNDNESS OF INDIAN BANKING <i>SAHILA CHAUDHARY &amp; DR. SULTAN SINGH</i>	26
6.	ASSURING QUALITY USING 6 SIGMA TOOL - DMAIC TECHNIQUE <i>ANOOP C NAIR</i>	34
7.	COMMUNITIES OF PRACTICE: THEIR ROLE IN THE CREATION AND TRANSFER OF KNOWLEDGE IN ORGANISATIONS <i>DR. ROOPA T.N. &amp; RAGHAVENDRA A.N.</i>	39
8.	MAMAGEMENT OF OVERALL PRODUCTIVITY IN SPOT WELDING CARRIED OUT IN WELD DIVISION OF A LIMITED COMPANY <i>DR. G RAJENDRA, AKSHATHA V. M &amp; HARSHA D</i>	43
9.	A STUDY ON THE PERFORMANCE OF INVENTORY MANAGEMENT IN APSRTC <i>DR. K. SAI KUMAR</i>	48
10.	IMPACT OF CHANGES IN ENTRY LOAD STRUCTURE OF MUTUAL FUND SCHEMES – EVIDENCE FROM INDIAN MUTUAL FUND INDUSTRY <i>N. VENKATESH KUMAR &amp; DR. ASHWINI KUMAR BJ</i>	56
11.	A COMPARATIVE ANALYSIS OF MARKET RETURNS AND FUND FLOWS WITH REFERENCE TO MUTUAL FUNDS <i>R. ANITHA, C. RADHAPRIYA &amp; T. DEVAENATHIPATHI</i>	62
12.	WOMEN EMPOWERMENT AND ENTREPRENEURSHIP THROUGH SHGs -A STUDY OF CHIKKABALLAPUR DISTRICT <i>DR. S. MURALIDHAR, K. SHARADA &amp; NARASAPPA.P.R</i>	67
13.	ANDHRA PRADESH STATE FINANCIAL CORPORATION FOR THE DEVELOPMENT OF MICRO, SMALL AND MEDIUM ENTERPRISES (MSMEs) - A STUDY OF TIRUPATI BRANCH IN CHITTOOR DISTRICT <i>DR. K. SUDARSAN, DR. V. MURALI KRISHNA, DR. KOTA SREENIVASA MURTHY &amp; DR. D. HIMACHALAM</i>	72
14.	IMPACT OF SERVICE QUALITY AND CUSTOMER SATISFACTION ON REPURCHASE INTENTION <i>ARUP KUMAR BAKSI &amp; DR. BIVRAJ BHUSAN PARIDA</i>	80
15.	AN EMPIRICAL RESEARCH ON MOBILE USERS INTENTION AND BEHAVIOUR TOWARDS MOBILE ENTERTAINMENT SERVICES IN INDIA BASED ON THEORY OF PLANNED BEHAVIOUR MODEL <i>G N SATISH KUMAR &amp; T. V. JANAKI</i>	86
16.	RETENTION STRATEGY: THE MAJOR TRENDS THAT CARRIED OUT IN IT SECTOR <i>DR. S. CHITRA DEVI &amp; E. LATHA</i>	90
17.	HUMAN RESOURCE DEVELOPMENT PRACTICES IN INFORMATION TECHNOLOGY INDUSTRY IN INDIA <i>DR. DEEPAKSHI GUPTA &amp; DR. NEENA MALHOTRA</i>	95
18.	ORGANISATIONAL SUPPORT FOR EMPLOYEES' CAREER MANAGEMENT <i>A. SEEMA, DR. ANITA PRIYA RAJA &amp; DR. S. SUJATHA</i>	109
19.	A STUDY ON SMALL INVESTOR'S PREFERENCE TOWARDS MUTUAL FUNDS IN SALEM DISTRICT, TAMIL NADU <i>M. GURUSAMY</i>	113
20.	A STUDY ON ATTRITION IN DOMESTIC FORMULATIONS IN CHENNAI CHEMICALS AND PHARMACEUTICALS LTD. <i>C M MARAN</i>	123
21.	A STATISTICAL ANALYSIS OF DAILY NIFTY RETURNS, DURING 2001-11 <i>SANTANU DUTTA</i>	133
22.	HEALTH AND SOCIAL PROBLEMS OF INDIAN WOMEN - A STUDY <i>DR. A. S. SHIRALASHETTI</i>	137
23.	ANTECEDENTS OF CRM IN HIGHER EDUCATION <i>DR. NARINDER TANWAR</i>	139
24.	HUMAN CAPITAL MANAGEMENT ISSUES AND POSSIBILITIES OF MSMEs - A STUDY ON SELECT UNITS IN BANGALORE <i>LAKSHMYPRIYA K. &amp; SUPARNA DAS PURKAYASTHA</i>	142
25.	GENERATING FUNCTIONS FOR PELL AND PELL-LUCAS NUMBERS <i>DR. NARESH PATEL</i>	152
	REQUEST FOR FEEDBACK	162

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- Sharma T., Kwatra, G. (2008) Effectiveness of Social Advertising: A Study of Selected Campaigns, Corporate Social Responsibility, Edited by David Crowther & Nicholas Capaldi, Ashgate Research Companion to Corporate Social Responsibility, Chapter 15, pp 287-303.

**JOURNAL AND OTHER ARTICLES**

- Schemenner, R.W., Huber, J.C. and Cook, R.L. (1987), "Geographic Differences and the Location of New Manufacturing Facilities," Journal of Urban Economics, Vol. 21, No. 1, pp. 83-104.

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**GENERATING FUNCTIONS FOR PELL AND PELL-LUCAS NUMBERS**

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**ABSTRACT**

In this paper, I have derived a list of generating functions for Pell and Pell-Lucas numbers. Exponential generating functions are used to derive combinatorial identities as well as hybrid identities. Generalizations of the results are the main features of this paper.

**KEYWORDS**

Exponential Functions, Generating Functions, Hybrid Identities.

**INTRODUCTION**

**PELL AND PELL-LUCAS NUMBERS**



Define the sequences  $\{U_n\}$  and  $\{V_n\}$  for all integers  $n$  by

$$\begin{cases} U_n = pU_{n-1} + U_{n-2}, & U_0 = 0, U_1 = 1, \\ V_n = pV_{n-1} + V_{n-2}, & V_0 = 2, V_1 = p. \end{cases}$$

For  $p = 1$ , we write  $\{U_n\} = \{F_n\}$  and  $\{V_n\} = \{L_n\}$ , which are the Fibonacci and Lucas numbers respectively. Their Binet forms, obtained by using

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad L_n = \alpha^n + \beta^n,$$

standard techniques for solving linear recurrences, are

where  $\alpha$  and  $\beta$  are the roots of  $x^2 - x - 1 = 0$ .

For  $p = 2$ , we write

$$\begin{cases} P_n = 2P_{n-1} + P_{n-2}, & P_0 = 0, P_1 = 1, \\ Q_n = 2Q_{n-1} + Q_{n-2}, & Q_0 = 2, Q_1 = 2. \end{cases}$$

Here  $\{P_n\}$  and  $\{Q_n\}$  are the Pell and Pell-Lucas Sequences respectively. Their Binet forms are given by

$$P_n = \frac{\gamma^n - \delta^n}{\gamma - \delta} \quad \text{and} \quad Q_n = \gamma^n + \delta^n,$$

where  $\gamma$  and  $\delta$  are the roots of  $x^2 - 2x - 1 = 0$  that is  $\gamma = 1 + \sqrt{2}$  and  $\delta = 1 - \sqrt{2}$ .

**GENERATING FUNCTIONS**

Let  $a_0, a_1, a_2, \dots$  be a sequence of real numbers. Then the function  $g(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots = \sum_{n=0}^{\infty} a_nx^n$  is called

the generating function for the sequence  $\{a_n\}$ . We can also define generating functions for the finite sequence  $a_0, a_1, \dots, a_n$  by letting  $a_i = 0$  for  $i > n$ ;

thus  $g(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$  is called the generating function for the finite sequence  $a_0, a_1, \dots, a_n$ .

For example,  $g(x) = 1 + 2x + 3x^2 + \dots + (n+1)x^n + \dots$  is the generating function for the sequence of positive integers and

$f(x) = 1 + 3x + 6x^2 + \dots + \frac{n(n+1)}{2}x^n + \dots$  is the generating function for the sequence of triangular numbers 1, 3, 6, 10, ....

Also  $g(x) = 1 + x + x^2 + \dots + x^n + \dots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x};$  (1)

Thus  $\left(\frac{1}{1-x}\right)$  is the generating function for the infinite sequence of ones, whereas

$$f(x) = \frac{x^n - 1}{x - 1} = 1 + x + x^2 + \dots + x^{n-1} \quad ; \text{ Thus } \left( \frac{x^n - 1}{x - 1} \right) \text{ is the generating for the finite sequence of } n \text{ ones.}$$

**EQUALITY OF GENERATING FUNCTIONS**

Two generating functions  $f(x) = \sum_{n=0}^{\infty} a_n x^n$  and  $g(x) = \sum_{n=0}^{\infty} b_n x^n$  are equal if  $a_n = b_n$  for  $n \geq 0$ ; that is

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n \Rightarrow a_n = b_n \text{ for } n \geq 0.$$

**ADDITION AND MULTIPLICATION OF GENERATING FUNCTIONS**

Let  $f(x) = \sum_{n=0}^{\infty} a_n x^n$  and  $g(x) = \sum_{n=0}^{\infty} b_n x^n$  be two generating functions. Then

$$f(x) + g(x) = \sum_{n=0}^{\infty} (a_n + b_n) x^n$$

$$f(x)g(x) = \sum_{n=0}^{\infty} \left( \sum_{i=0}^n a_i b_{n-i} \right) x^n \tag{2}$$

**GENERATING FUNCTION FOR  $P_n$ :**

First, we develop a generating function  $g(x)$  for  $P_n$ . So, let  $g(x) = \sum_{n=0}^{\infty} P_n x^n$  ;

$$\begin{aligned} g(x) &= P_0 + P_1 x + P_2 x^2 + \dots + P_n x^n + P_{n+1} x^{n+1} + \dots \\ 2xg(x) &= 2P_0 x + 2P_1 x^2 + \dots + 2P_{n-1} x^n + 2P_n x^{n+1} + \dots \\ x^2 g(x) &= P_0 x^2 + \dots + P_{n-2} x^n + P_{n-1} x^{n+1} + \dots \end{aligned}$$

$$\therefore (1 - 2x - x^2)g(x) = P_0 + P_1 x - 2P_0 x = 2x \quad (\text{Since } P_{n+1} = 2P_n + P_{n-1})$$

Using  $P_0 = 0$  and  $P_1 = 1$ , we get

$$\sum_{n=0}^{\infty} P_n x^n = \frac{x}{(1 - 2x - x^2)} \tag{3}$$

Therefore,  $\frac{x}{(1 - 2x - x^2)}$  generates the sequence of Pell numbers.

**GENERATING FUNCTION FOR  $P_{n+1}P_{n+2}$ :**

Let  $g(x) = \sum_{n=0}^{\infty} P_{n+1}P_{n+2}x^n$  ; so that

$$\begin{aligned} g(x) &= P_1 P_2 + P_2 P_3 x + P_3 P_4 x^2 + P_4 P_5 x^3 + \dots + P_{n+1} P_{n+2} x^n + P_{n+2} P_{n+3} x^{n+1} + \dots \\ 5xg(x) &= 5P_1 P_2 x + 5P_2 P_3 x^2 + 5P_3 P_4 x^3 + \dots + 5P_n P_{n+1} x^n + 5P_{n+1} P_{n+2} x^{n+1} + \dots \\ 5x^2 g(x) &= 5P_1 P_2 x^2 + 5P_2 P_3 x^3 + \dots + 5P_{n-1} P_n x^n + 5P_n P_{n+1} x^{n+1} + \dots \\ x^3 g(x) &= P_1 P_2 x^3 + \dots + P_{n-2} P_{n-1} x^n + P_{n-1} P_n x^{n+1} + \dots \end{aligned}$$

$$\begin{aligned} \therefore (1 - 5x - 5x^2 + x^3)g(x) &= (P_1 P_2 + P_2 P_3 x + P_3 P_4 x^2) - (5P_1 P_2 x + 5P_2 P_3 x^2) - (5P_1 P_2 x^2) \\ &= (2 + 10x + 60x^2) - (10x + 50x^2) - (10x^2) \\ &= (2 + 10x + 60x^2) - (10x + 50x^2) - (10x^2) \end{aligned}$$

$$\text{Thus, } \sum_{n=0}^{\infty} P_{n+1}P_{n+2}x^n = \frac{2}{(1-5x-5x^2+x^3)}$$

**GENERATING FUNCTION FOR  $P_n^2$  :**

$$g(x) = \sum_{n=0}^{\infty} P_n^2 x^n$$

Let ; so that

$$g(x) = P_0^2 + P_1^2 x + P_2^2 x^2 + P_3^2 x^3 + P_4^2 x^4 + \dots + P_n^2 x^n + P_{n+1}^2 x^{n+1} + \dots$$

$$5xg(x) = 5P_0^2 x + 5P_1^2 x^2 + 5P_2^2 x^3 + 5P_3^2 x^4 + \dots + 5P_{n-1}^2 x^n + 5P_n^2 x^{n+1} + \dots$$

$$5x^2g(x) = 5P_0^2 x^2 + 5P_1^2 x^3 + 5P_2^2 x^4 + \dots + 5P_{n-2}^2 x^n + 5P_{n-1}^2 x^{n+1} + \dots$$

$$x^3g(x) = P_0^2 x^3 + P_1^2 x^4 + \dots + P_{n-3}^2 x^n + P_{n-2}^2 x^{n+1} + \dots$$

$$\begin{aligned} \therefore (1-5x-5x^2+x^3)g(x) &= (P_0^2 + P_1^2 x + P_2^2 x^2) - (5P_0^2 x + 5P_1^2 x^2) - (5P_0^2 x^2) \\ &\quad \left[ \because P_{n+1}^2 + P_{n+2}^2 = 5(P_n^2 + P_{n+1}^2) \right] \\ &= (0 + x + 4x^2) - (0x + 5x^2) - (0x^2) = x(1-x) \end{aligned}$$

$$\text{Thus, } \sum_{n=0}^{\infty} P_n^2 x^n = \frac{x(1-x)}{(1-5x-5x^2+x^3)}$$

**GENERATING FUNCTION FOR  $P_n^3$  :**

$$g(x) = \sum_{n=0}^{\infty} P_n^3 x^n$$

Let ; so that

$$g(x) = P_0^3 + P_1^3 x + P_2^3 x^2 + P_3^3 x^3 + P_4^3 x^4 + P_5^3 x^5 + \dots + P_n^3 x^n + P_{n+1}^3 x^{n+1} + \dots$$

$$12xg(x) = 12P_0^3 x + 12P_1^3 x^2 + 12P_2^3 x^3 + 12P_3^3 x^4 + 12P_4^3 x^5 + \dots + 12P_{n-1}^3 x^n + 12P_n^3 x^{n+1} + \dots$$

$$30x^2g(x) = 30P_0^3 x^2 + 30P_1^3 x^3 + 30P_2^3 x^4 + 30P_3^3 x^5 + \dots + 30P_{n-2}^3 x^n + 30P_{n-1}^3 x^{n+1} + \dots$$

$$12x^3g(x) = 12P_0^3 x^3 + 12P_1^3 x^4 + 12P_2^3 x^5 + \dots + 12P_{n-3}^3 x^n + 12P_{n-2}^3 x^{n+1} + \dots$$

$$x^4g(x) = P_0^3 x^4 + P_1^3 x^5 + \dots + P_{n-4}^3 x^n + P_{n-3}^3 x^{n+1} + \dots$$

$$\begin{aligned} \therefore (1-12x-30x^2+12x^3+x^4)g(x) &= (P_0^3 + P_1^3 x + P_2^3 x^2 + P_3^3 x^3) - (12P_0^3 x + 12P_1^3 x^2 + 12P_2^3 x^2) \\ &\quad - (30P_0^3 x^2 + 30P_1^3 x^3) + (12P_0^3 x^3) \end{aligned}$$

$$\begin{aligned} &\quad \left[ \because P_{n+4}^3 - 12P_{n+3}^3 - 30P_{n+2}^3 + 12P_{n+1}^3 + P_n^3 = 0 \right] \\ &= (0 + x + 8x^2 + 125x^3) - (0x + 12x^2 + 96x^3) - (0x^2 + 30x^3) + (0x^3) \\ &= x - 4x^2 - x^3 \end{aligned}$$

$$\text{Thus, } \sum_{n=0}^{\infty} P_n^3 x^n = \frac{x(1-4x-x^2)}{(1-12x-30x^2+12x^3+x^4)}$$

**A LIST OF GENERATING FUNCTIONS**

Using the above technique and Pell Identities, we can also derive the following generating functions:

$$(i) \frac{1-2x}{(1-2x-x^2)} = \sum_{n=0}^{\infty} P_{n-1} x^n \tag{4}$$

$$(ii) \frac{1}{(1-2x-x^2)} = \sum_{n=0}^{\infty} P_{n+1} x^n \tag{5}$$



$$(iii) \frac{2(1-x)}{(1-2x-x^2)} = \sum_{n=0}^{\infty} Q_n x^n \tag{6}$$

$$(iv) \frac{2(1+x)}{(1-2x-x^2)} = \sum_{n=0}^{\infty} Q_{n+1} x^n \tag{7}$$

$$(v) \frac{-2+6x}{(1-2x-x^2)} = \sum_{n=0}^{\infty} Q_{n-1} x^n$$

$$(vi) \frac{2x}{(1-6x+x^2)} = \sum_{n=0}^{\infty} P_{2n} x^n \quad (\text{Using } P_{2n+2} + P_{2n-2} = 6P_{2n}, n \geq 1)$$

$$(vii) \frac{1-x}{(1-6x+x^2)} = \sum_{n=0}^{\infty} P_{2n+1} x^n \quad (\text{Using } P_{2n+3} + P_{2n-1} = 6P_{2n+1}, n \geq 1)$$

$$(viii) \frac{2(1-3x)}{(1-6x+x^2)} = \sum_{n=0}^{\infty} Q_{2n} x^n \quad (\text{Using } Q_{2n+2} + Q_{2n-2} = 6Q_{2n}, n \geq 1)$$

$$(ix) \frac{2(1+x)}{(1-6x+x^2)} = \sum_{n=0}^{\infty} Q_{2n+1} x^n \quad (\text{Using } Q_{2n+3} + Q_{2n-1} = 6Q_{2n+1}, n \geq 1)$$

$$(x) \frac{2(3-x)}{(1-6x+x^2)} = \sum_{n=0}^{\infty} Q_{2n+2} x^n$$

$$(xi) \frac{2}{(1-6x+x^2)} = \sum_{n=0}^{\infty} P_{2n+2} x^n$$

$$(xii) \frac{5x}{(1-14x-x^2)} = \sum_{n=0}^{\infty} P_{3n} x^n \quad (\text{Using } P_{3n+3} - P_{3n-3} = 14P_{3n}, n \geq 1)$$

$$(xiii) \frac{2-14x}{(1-14x-x^2)} = \sum_{n=0}^{\infty} Q_{3n} x^n \quad (\text{Using } Q_{3n+3} - Q_{3n-3} = 14Q_{3n}, n \geq 1)$$

$$(xiv) \frac{1-x}{(1-5x-5x^2+x^3)} = \sum_{n=0}^{\infty} P_{n+1}^2 x^n$$

$$(xv) \frac{4+5x-x^2}{(1-5x-5x^2+x^3)} = \sum_{n=0}^{\infty} P_{n+2}^2 x^n$$

$$(xvi) \frac{4-16x-4x^2}{(1-5x-5x^2+x^3)} = \sum_{n=0}^{\infty} Q_n^2 x^n$$

$$(xvii) \frac{4+16x-4x^2}{(1-5x-5x^2+x^3)} = \sum_{n=0}^{\infty} Q_{n+1}^2 x^n$$

$$(xviii) \frac{36+16x-4x^2}{(1-5x-5x^2+x^3)} = \sum_{n=0}^{\infty} Q_{n+2}^2 x^n$$

$$(xix) \frac{8-88x-120x^2+8x^3}{(1-12x-30x^2+12x^3+x^4)} = \sum_{n=0}^{\infty} Q_n^3 x^n$$

$$(xx) \frac{1-4x-x^2}{(1-12x-30x^2+12x^3+x^4)} = \sum_{n=0}^{\infty} P_{n+1}^3 x^n$$

$$(xxi) \frac{8+120x-88x^2-8x^3}{(1-12x-30x^2+12x^3+x^4)} = \sum_{n=0}^{\infty} Q_{n+1}^3 x^n$$

$$(xxii) \frac{8 + 29x - 12x^2 - x^3}{(1 - 12x - 30x^2 + 12x^3 + x^4)} = \sum_{n=0}^{\infty} P_{n+2}^3 x^n$$

$$(xxiii) \frac{216 + 152x - 104x^2 - 8x^3}{(1 - 12x - 30x^2 + 12x^3 + x^4)} = \sum_{n=0}^{\infty} Q_{n+2}^3 x^n$$

$$(xxiv) \frac{10x}{(1 - 12x - 30x^2 + 12x^3 + x^4)} = \sum_{n=0}^{\infty} P_n P_{n+1} P_{n+2} x^n$$

$$(xxv) \frac{24 - 120x + 120x^2 + 8x^3}{(1 - 12x - 30x^2 + 12x^3 + x^4)} = \sum_{n=0}^{\infty} Q_n Q_{n+1} Q_{n+2} x^n$$

$$(xxvi) \frac{P_k x}{1 - Q_k x + (-1)^k x^2} = \sum_{n=0}^{\infty} P_{nk} x^n$$

$$(xxvii) \frac{P_r + (-1)^r P_{k-r} x}{1 - Q_k x + (-1)^k x^2} = \sum_{n=0}^{\infty} P_{nk+r} x^n$$

$$(xxviii) \frac{2 + 3Q_k x}{1 - Q_k x + (-1)^k x^2} = \sum_{n=0}^{\infty} Q_{nk} x^n$$

$$(xxvii) \frac{Q_r + (-1)^r Q_{k-r} x}{1 - Q_k x + (-1)^k x^2} = \sum_{n=0}^{\infty} Q_{nk+r} x^n$$

**GENERATING FUNCTIONS FOR  $S_n$  :**

Let  $s_n = \sum_{i=1}^n P_i$  and  $g(x) = \sum_{n=0}^{\infty} s_n x^n$  ; then

$$\begin{aligned} \sum_{n=0}^{\infty} s_n x^n &= \sum_{n=0}^{\infty} \frac{1}{2} (P_n + P_{n+1} - 1) x^n && \sum_{i=1}^n P_i = \frac{1}{2} (P_n + P_{n+1} - 1) \\ &= \frac{1}{2} \left[ \sum_{n=0}^{\infty} P_n x^n + \sum_{n=0}^{\infty} P_{n+1} x^n - \sum_{n=0}^{\infty} x^n \right] && \text{[Since } \sum_{i=1}^n P_i = \frac{1}{2} (P_n + P_{n+1} - 1) \text{]} \\ &= \frac{1}{2} \left[ \frac{x}{(1 - 2x - x^2)} + \frac{1}{(1 - 2x - x^2)} - \frac{1}{1 - x} \right] && \text{[Using (1), (3) and (5)]} \\ &= \frac{1}{2} \left[ \frac{2x}{(1 - 3x + x^2 + x^3)} \right] \end{aligned}$$

Thus,  $\sum_{n=0}^{\infty} s_n x^n = \frac{x}{(1 - 3x + x^2 + x^3)}$ .

**GENERATING FUNCTIONS FOR  $P_{m+n}$  and  $Q_{m+n}$  :**

Using Binet formula, generating function for  $P_{m+n}$  can be derived as follows:

$$\begin{aligned} \sum_{n=0}^{\infty} P_{m+n} x^n &= \sum_{n=0}^{\infty} \left( \frac{\gamma^{m+n} - \delta^{m+n}}{\gamma - \delta} \right) x^n \\ &= \frac{1}{\gamma - \delta} \left[ \gamma^m \sum_{n=0}^{\infty} \gamma^n x^n - \delta^m \sum_{n=0}^{\infty} \delta^n x^n \right] \\ &= \frac{1}{\gamma - \delta} \left[ \frac{\gamma^m}{1 - \gamma x} - \frac{\delta^m}{1 - \delta x} \right] && \sum_{n=0}^{\infty} a^n x^n = \frac{1}{1 - ax} \text{ [Since]} \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{\gamma - \delta} \left[ \frac{\gamma^m (1 - \delta x) - \delta^m (1 - \gamma x)}{(1 - \gamma x)(1 - \delta x)} \right] \\
 &= \frac{1}{\gamma - \delta} \left[ \frac{(\gamma^m - \delta^m) - \gamma \delta x (\gamma^{m-1} - \delta^{m-1})}{1 - (\gamma + \delta)x + \gamma \delta x^2} \right] \\
 &= \frac{1}{\gamma - \delta} \left[ \frac{(\gamma^m - \delta^m) + (\gamma^{m-1} - \delta^{m-1})x}{1 - (\gamma + \delta)x + \gamma \delta x^2} \right] \\
 &= \frac{P_m + P_{m-1}x}{1 - 2x - x^2} \\
 \text{Thus, } \sum_{n=0}^{\infty} P_{m+n} x^n &= \left( \frac{P_m + P_{m-1}x}{1 - 2x - x^2} \right).
 \end{aligned}$$

Likewise, it can be shown that  $\sum_{n=0}^{\infty} Q_{m+n} x^n = \left( \frac{Q_m + Q_{m-1}x}{1 - 2x - x^2} \right)$ .

**EXPONENTIAL GENERATING FUNCTIONS**

Since  $e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}$ , it follows that  $e^{\gamma x} = \sum_{n=0}^{\infty} \frac{\gamma^n x^n}{n!}$  and  $e^{\delta x} = \sum_{n=0}^{\infty} \frac{\delta^n x^n}{n!}$

$$\therefore \frac{e^{\gamma x} - e^{\delta x}}{\gamma - \delta} = \sum_{n=0}^{\infty} \left( \frac{\gamma^n - \delta^n}{\gamma - \delta} \right) \frac{x^n}{n!} = \sum_{n=0}^{\infty} P_n \frac{x^n}{n!} \tag{8}$$

Thus, the exponential function  $\frac{e^{\gamma x} - e^{\delta x}}{\gamma - \delta}$  generates the numbers  $\frac{P_n}{n!}$ .

Likewise, we can show that

$$e^{\gamma x} + e^{\delta x} = \sum_{n=0}^{\infty} Q_n \frac{x^n}{n!}; \tag{9}$$

That is  $(e^{\gamma x} + e^{\delta x})$  generates the numbers  $\frac{Q_n}{n!}$ .

More generally, we can show that

$$\frac{e^{\gamma^k x} - e^{\delta^k x}}{\gamma - \delta} = \sum_{n=0}^{\infty} \frac{P_{nk}}{n!} x^n \tag{10}$$

Likewise,

$$e^{\gamma^k x} + e^{\delta^k x} = \sum_{n=0}^{\infty} \frac{Q_{nk}}{n!} x^n \tag{11}$$

Using (8), we can write

$$\frac{e^{(1+\sqrt{2})x} - e^{(1-\sqrt{2})x}}{(1+\sqrt{2}) - (1-\sqrt{2})} = \sum_{n=0}^{\infty} P_n \frac{x^n}{n!}$$

Or  $e^{(1+\sqrt{2})x} - e^{(1-\sqrt{2})x} = 2\sqrt{2} \sum_{n=0}^{\infty} P_n \frac{x^n}{n!}$

Or  $e^x [e^{\sqrt{2}x} - e^{-\sqrt{2}x}] = 2\sqrt{2} \sum_{n=0}^{\infty} P_n \frac{x^n}{n!}$

Or  $2e^x \sinh \sqrt{2}x = 2\sqrt{2} \sum_{n=0}^{\infty} P_n \frac{x^n}{n!}$

Thus,

$$e^x \sinh \sqrt{2}x = \sqrt{2} \sum_{n=0}^{\infty} P_n \frac{x^n}{n!}$$

Similarly,

$$2e^x \cosh \sqrt{2}x = \sum_{n=0}^{\infty} Q_n \frac{x^n}{n!}$$

**IDENTITIES USING GENERATING FUNCTIONS**

We have  $\sum_{n=0}^{\infty} P_{n+1}x^n = \frac{1}{D}$ ,  $\sum_{n=0}^{\infty} P_{n-1}x^n = \frac{1-2x}{D}$  and  $\sum_{n=0}^{\infty} Q_n x^n = \frac{2(1-x)}{D}$  where  $D = 1 - 2x - x^2$ . Also,

$$\frac{2(1-x)}{D} = \frac{1}{D} + \frac{1-2x}{D}$$

It follows that  $\sum_{n=0}^{\infty} \frac{2(1-x)}{D} = \sum_{n=0}^{\infty} \frac{1}{D} + \sum_{n=0}^{\infty} \frac{1-2x}{D}$

$$\Rightarrow \sum_{n=0}^{\infty} Q_n x^n = \sum_{n=0}^{\infty} P_{n+1}x^n + \sum_{n=0}^{\infty} P_{n-1}x^n$$

$$= \sum_{n=0}^{\infty} (P_{n+1} + P_{n-1})x^n$$

$$\therefore Q_n = P_{n+1} + P_{n-1}$$

Next we shall prove that  $P_m Q_n + P_{m-1} Q_{n-1} = Q_{m+n-1}$ .

$$\sum_{m=0}^{\infty} (P_m Q_n + P_{m-1} Q_{n-1})x^m = Q_n \sum_{m=0}^{\infty} P_m x^m + Q_{n-1} \sum_{m=0}^{\infty} P_{m-1} x^m$$

$$= Q_n \frac{x}{D} + Q_{n-1} \frac{1-2x}{D}$$

$$= \frac{Q_{n-1} + (Q_n - 2Q_{n-1})x}{D} = \frac{Q_{n-1} + Q_{n-2}x}{D}$$

$$= \sum_{m=0}^{\infty} Q_{m+n-1} x^m$$

$$\therefore P_m Q_n + P_{m-1} Q_{n-1} = Q_{m+n-1}$$

Similarly, we can prove other identities like

$$P_m P_n + P_{m-1} P_{n-1} = P_{m+n-1}, Q_m Q_n + Q_{m-1} Q_{n-1} = 8P_{m+n-1} \text{ etc.}$$

**COMBINATORIAL IDENTITIES USING GENERATING FUNCTIONS**

Let  $A(t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!}$  and  $B(t) = \sum_{n=0}^{\infty} b_n \frac{t^n}{n!}$ ; then

$$A(t)B(t) = \sum_{n=0}^{\infty} \left[ \sum_{k=0}^n \binom{n}{k} a_k b_{n-k} \right] \frac{t^n}{n!} \tag{12}$$

$$A(t)B(-t) = \sum_{n=0}^{\infty} \left[ \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} a_k b_{n-k} \right] \frac{t^n}{n!} \tag{13}$$

and  $A(2t)B(t) = \sum_{n=0}^{\infty} \left[ \sum_{k=0}^n 2^k \binom{n}{k} a_k b_{n-k} \right] \frac{t^n}{n!}$  (14)

In particular,  $A(t) = \frac{e^{\gamma t} - e^{\delta t}}{\gamma - \delta}$  and  $B(t) = e^t$ ; then

$$A(2t)B(t) = \left( \frac{e^{2\gamma t} - e^{2\delta t}}{\gamma - \delta} \right) e^t$$

$$\sum_{n=0}^{\infty} \left[ \sum_{k=0}^n 2^k \binom{n}{k} a_k b_{n-k} \right] \frac{t^n}{n!} = \left( \frac{e^{2\gamma t} - e^{2\delta t}}{\gamma - \delta} \right) e^t$$

By (14)

$$\sum_{n=0}^{\infty} \left[ \sum_{k=0}^n 2^k \binom{n}{k} P_k \right] \frac{t^n}{n!} = \left( \frac{e^{(1+2\gamma)t} - e^{(1+2\delta)t}}{\gamma - \delta} \right)$$

By (8)

$$= \left( \frac{e^{\gamma^2 t} - e^{\delta^2 t}}{\gamma - \delta} \right)$$

(Since  $\gamma^2 - 2\gamma - 1 = 0$ ),

$$= \sum_{n=0}^{\infty} P_{2n} \frac{t^n}{n!}$$

By (10)

Thus,

$$\sum_{n=0}^{\infty} \left[ \sum_{k=0}^n 2^k \binom{n}{k} P_k \right] \frac{t^n}{n!} = \sum_{n=0}^{\infty} P_{2n} \frac{t^n}{n!}$$

Equating the coefficients of  $\frac{t^n}{n!}$  yields the combinatorial identity

$$\sum_{k=0}^n 2^k \binom{n}{k} P_k = P_{2n}$$

Now taking,

$A(t) = \frac{e^{\gamma t} - e^{\delta t}}{\gamma - \delta}$  and  $B(t) = e^{-t}$ ;

$$A(t)B(2t) = \left( \frac{e^{\gamma t} - e^{\delta t}}{\gamma - \delta} \right) e^{-2t}$$

$$\sum_{n=0}^{\infty} \left[ \sum_{k=0}^n (-2)^{n-k} \binom{n}{k} a_k b_{n-k} \right] \frac{t^n}{n!} = \left( \frac{e^{\gamma t} - e^{\delta t}}{\gamma - \delta} \right) e^{-2t}$$

$$\sum_{n=0}^{\infty} \left[ \sum_{k=0}^n (-2)^{n-k} \binom{n}{k} P_k \right] \frac{t^n}{n!} = \left( \frac{e^{(-2+\gamma)t} - e^{(-2+\delta)t}}{\gamma - \delta} \right)$$

$$= \left( \frac{e^{-\delta t} - e^{-\gamma t}}{\gamma - \delta} \right)$$

(Since  $\gamma + \delta = 2$ ),

$$= \sum_{n=0}^{\infty} (-1) P_n \frac{(-t)^n}{n!} = \sum_{n=0}^{\infty} (-1)^{n-1} P_n \frac{t^n}{n!}$$

Thus,

$$\sum_{n=0}^{\infty} \left[ \sum_{k=0}^n (-2)^{n-k} \binom{n}{k} P_k \right] \frac{t^n}{n!} = \sum_{n=0}^{\infty} (-1)^{n-1} P_n \frac{t^n}{n!}$$

Equating the coefficients of  $\frac{t^n}{n!}$  yields another combinatorial identity

$$\sum_{k=0}^n (-2)^{n-k} \binom{n}{k} P_k = (-1)^{n-1} P_n$$

Obviously, by selecting  $A(t)$  and  $B(t)$  as suitable exponential functions, we can apply this method to derive other Pell and Pell-Lucas combinatorial identities.

For example, choosing  $A(t) = e^{\gamma t} + e^{\delta t}$  and  $B(t) = e^t$ , we get  $\sum_{k=0}^n 2^k \binom{n}{k} Q_k = Q_{2n}$ ; and taking  $A(t) = e^{\gamma t} + e^{\delta t}$  and  $B(t) = e^{-t}$ , we get  $\sum_{k=0}^n (-2)^{n-k} \binom{n}{k} Q_k = (-1)^n Q_n$ .

**COMBINATORIAL IDENTITIES USING THE DIFFERENTIAL OPERATOR**

Since  $A(t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!}$ , it follows that  $\frac{d^r}{dt^r} A(t) = \sum_{n=0}^{\infty} a_{n+r} \frac{t^n}{n!}$  (15)

In particular, taking  $A(t) = \frac{e^{2\alpha t} - e^{2\beta t}}{\alpha - \beta}$  and  $B(t) = e^t$ ; we get

$$A(t)B(t) = \left( \frac{e^{2\alpha t} - e^{2\beta t}}{\alpha - \beta} \right) e^t$$

$$A(t)B(t) = \sum_{n=0}^{\infty} \left[ \sum_{k=0}^n 2^k \binom{n}{k} P_k \right] \frac{t^n}{n!}$$

$$\left( \because A(t)B(t) = \sum_{n=0}^{\infty} \left[ \sum_{k=0}^n \binom{n}{k} a_k b_{n-k} \right] \frac{t^n}{n!} \right)$$

Using (15) we can write

$$B(t) \frac{d^r}{dt^r} A(t) = \sum_{n=0}^{\infty} \left[ \sum_{k=0}^n 2^{k+r} \binom{n}{k} P_{k+r} \right] \frac{t^n}{n!}$$

Or  $\sum_{n=0}^{\infty} \left[ \sum_{k=0}^n 2^{k+r} \binom{n}{k} P_{k+r} \right] \frac{t^n}{n!} = B(t) \frac{d^r}{dt^r} \left( \frac{e^{2\alpha t} - e^{2\beta t}}{\alpha - \beta} \right)$

$$= e^t \cdot \left[ \frac{(2\alpha)^r e^{2\alpha t} - (2\beta)^r e^{2\beta t}}{\alpha - \beta} \right]$$

$$= \frac{(2\alpha)^r e^{(2\alpha+1)t} - (2\beta)^r e^{(2\beta+1)t}}{\alpha - \beta}$$

$$= 2^r \left( \frac{\alpha^r e^{\alpha t} - \beta^r e^{\beta t}}{\alpha - \beta} \right) \quad (\text{Since } \alpha^2 - 2\alpha - 1 = 0)$$

$$= 2^r \sum_{n=0}^{\infty} P_{2n+r} \frac{t^n}{n!}$$

Thus,

$$\sum_{n=0}^{\infty} \left[ \sum_{k=0}^n 2^{k+r} \binom{n}{k} P_{k+r} \right] \frac{t^n}{n!} = 2^r \sum_{n=0}^{\infty} P_{2n+r} \frac{t^n}{n!}$$

Equating the coefficients of  $\frac{t^n}{n!}$  yields the combinatorial identity

$$\sum_{k=0}^n 2^{k+r} \binom{n}{k} P_{k+r} = 2^r P_{2n+r}.$$

Or  $\sum_{k=0}^n 2^k \binom{n}{k} P_{k+r} = P_{2n+r}.$

HYBRID IDENTITIES

$$A(t) = \frac{e^{\gamma t} - e^{\delta t}}{\gamma - \delta} \text{ and } B(t) = e^{\gamma t} + e^{\delta t}; \text{ then}$$

Here we derive some hybrid identities involving both Pell and Pell-Lucas numbers. If we choose

$$A(t)B(t) = \left( \frac{e^{\gamma t} - e^{\delta t}}{\gamma - \delta} \right) (e^{\gamma t} + e^{\delta t})$$

Or 
$$\sum_{n=0}^{\infty} \left[ \sum_{k=0}^n \binom{n}{k} P_k Q_{n-k} \right] \frac{t^n}{n!} = \left( \frac{e^{2\gamma t} - e^{2\delta t}}{\gamma - \delta} \right)$$

[By (8) and (12)]

Or 
$$\sum_{n=0}^{\infty} \left[ \sum_{k=0}^n \binom{n}{k} P_k Q_{n-k} \right] \frac{t^n}{n!} = \sum_{n=0}^{\infty} P_n \frac{(2t)^n}{n!}$$

Or 
$$\sum_{n=0}^{\infty} \left[ \sum_{k=0}^n \binom{n}{k} P_k Q_{n-k} \right] \frac{t^n}{n!} = \sum_{n=0}^{\infty} 2^n P_n \frac{t^n}{n!}$$

This yields the combinatorial hybrid identity that

$$\sum_{k=0}^n \binom{n}{k} P_k Q_{n-k} = 2^n P_n \tag{16}$$

$$A(t) = \frac{e^{\gamma t} - e^{\delta t}}{\gamma - \delta} = B(t)$$

By choosing  $t = 1$ , we get

$$A(1)B(1) = \left( \frac{e^{\gamma} - e^{\delta}}{\gamma - \delta} \right) (e^{\gamma} + e^{\delta})$$

Or 
$$\sum_{n=0}^{\infty} \left[ \sum_{k=0}^n \binom{n}{k} P_k P_{n-k} \right] \frac{t^n}{n!} = \left( \frac{e^{2\gamma} + e^{2\delta} - 2e^{(\gamma+\delta)}}{(\gamma - \delta)^2} \right)$$

$$\begin{aligned} &= \frac{1}{8} [(e^{2\gamma} + e^{2\delta}) - 2e^{2t}] \\ &= \sum_{n=0}^{\infty} \frac{1}{8} [2^n Q_n - 2 \cdot 2^n] \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} 2^{n-3} (Q_n - 2) \frac{t^n}{n!} \end{aligned}$$

Thus, 
$$\sum_{k=0}^n \binom{n}{k} P_k P_{n-k} = 2^{n-3} (Q_n - 2)$$

(17)

Similarly, 
$$\sum_{k=0}^n \binom{n}{k} Q_k Q_{n-k} = 2^n (Q_n + 2)$$

(18)

GENERALIZED HYBRID IDENTITIES

Identities (16) through (18) can also be generalized as follows:

$$\begin{aligned} \sum_{k=0}^n \binom{n}{k} P_{mk} Q_{mn-mk} &= 2^n P_{mn} \\ \sum_{k=0}^n \binom{n}{k} P_{mk} P_{mn-mk} &= \frac{2^n Q_{mn} - 2Q_m^n}{8} \\ \sum_{k=0}^n \binom{n}{k} Q_{mk} Q_{mn-mk} &= 2^n Q_{mn} + 2Q_m^n \end{aligned}$$

REFERENCES

[1] Brook, M. "Fibonacci Formulas", *Fibonacci Quarterly* 1, 60, 1963.  
 [2] Horadam A.F. "Pell Identities", *Fibonacci Quarterly* 9(3): 245-253, 263, 1971.  
 [3] Koshy, T. *Fibonacci and Lucas Numbers with Applications*. New York: Wiley, 2001.  
 [4] Santana, S. F. and Diaz-Barrero, J. L. "Some Properties of Sums Involving Pell Number". *Missouri Journal of Mathematical Sciences* 18 (1), 2006.

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Sd/-

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