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CONTRIBUTIONS TO BOOKS

- Sharma T., Kwatra, G. (2008) Effectiveness of Social Advertising: A Study of Selected Campaigns, Corporate Social Responsibility, Edited by David Crowther & Nicholas Capaldi, Ashgate Research Companion to Corporate Social Responsibility, Chapter 15, pp 287-303.

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- Schemenner, R.W., Huber, J.C. and Cook, R.L. (1987), "Geographic Differences and the Location of New Manufacturing Facilities," Journal of Urban Economics, Vol. 21, No. 1, pp. 83-104.

CONFERENCE PAPERS

- Garg Sambhav (2011): "Business Ethics" Paper presented at the Annual International Conference for the All India Management Association, New Delhi, India, 19–22 June.

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- Kumar S. (2011): "Customer Value: A Comparative Study of Rural and Urban Customers," Thesis, Kurukshetra University, Kurukshetra.

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WEBSITE

- Garg, Bhavet (2011): Towards a New Natural Gas Policy, Economic and Political Weekly, Viewed on July 05, 2011 http://epw.in/user/viewabstract.jsp

GENERATING FUNCTIONS FOR PELL AND PELL-LUCAS NUMBERS

DR. NARESH PATEL
ASST. PROFESSOR
DEPARTMENT OF APPLIED SCIENCES
INSTITUTE OF ENGINEERING & TECHNOLOGY
DEVI AHILYA UNIVERSITY
INDORE – 452 017

ABSTRACT

In this paper, I have derived a list of generating functions for Pell and Pell-Lucas numbers. Exponential generating functions are used to derive combinatorial identities as well as hybrid identities. Generalizations of the results are the main features of this paper.

KEYWORDS
Exponential Functions, Generating Functions, Hybrid Identities.

INTRODUCTION

PELL AND PELL-LUCAS NUMBERS

Define the sequences $\{U_n\}$ and $\{V_n\}$ for all integers *n* by

$$\begin{cases} U_n = pU_{n-1} + U_{n-2}, & U_0 = 0,\ U_1 = 1, \\ V_n = pV_{n-1} + V_{n-2}, & V_0 = 2,\ V_1 = p. \end{cases}$$

For *p* = 1, we write $\{U_n\} = \{F_n\}$ and $\{V_n\} = \{L_n\}$, which are the Fibonacci and Lucas numbers respectively. Their Binet forms, obtained by using standard techniques for solving linear recurrences, are $F_n = \dfrac{\alpha^n - \beta^n}{\alpha - \beta}$ and $L_n = \alpha^n + \beta^n$,

where α and β are the roots of $x^2 - x - 1 = 0$.

For *p* = 2, we write

$$\begin{cases} P_n = 2P_{n-1} + P_{n-2}, & P_0 = 0,\ P_1 = 1, \\ Q_n = 2Q_{n-1} + Q_{n-2}, & Q_0 = 2,\ Q_1 = 2. \end{cases}$$

Here $\{P_n\}$ and $\{Q_n\}$ are the Pell and Pell-Lucas Sequences respectively. Their Binet forms are given by

$$P_n = \frac{\gamma^n - \delta^n}{\gamma - \delta} \quad \text{and} \quad Q_n = \gamma^n + \delta^n,$$

where γ and δ are the roots of $x^2 - 2x - 1 = 0$ that is $\gamma = 1 + \sqrt{2}$ and $\delta = 1 - \sqrt{2}$.

GENERATING FUNCTIONS

Let $a_0, a_1, a_2, \ldots$ be a sequence of real numbers. Then the function $g(x) = a_0 + a_1 x + a_2 x^2 + \ldots. + a_n x^n + \ldots. = \sum_{n=0}^{\infty} a_n x^n$ is called the generating function for the sequence $\{a_n\}$. We can also define generating functions for the finite sequence $a_0, a_1, \ldots, a_n$ by letting $a_i = 0$ for $i > n$; thus $g(x) = a_0 + a_1 x + a_2 x^2 + \ldots. + a_n x^n$ is called the generating function for the finite sequence $a_0, a_1, \ldots, a_n$.

For example, $g(x) = 1 + 2x + 3x^2 + \ldots. + (n+1)x^n + \ldots.$ is the generating function for the sequence of positive integers and $f(x) = 1 + 3x + 6x^2 + \ldots. + \dfrac{n(n+1)}{2} x^n + \ldots.$ is the generating function for the sequence of triangular numbers 1, 3, 6, 10, ….

Also $g(x) = 1 + x + x^2 + \ldots. + x^n + \ldots. = \sum_{n=0}^{\infty} x^n = \dfrac{1}{1-x}$; (1)

Thus $\left(\dfrac{1}{1-x}\right)$ is the generating function for the infinite sequence of ones, whereas

$f(x)=\dfrac{x^n-1}{x-1}=1+x+x^2+....+x^{n-1}$; Thus $\left(\dfrac{x^n-1}{x-1}\right)$ is the generating for the finite sequence of *n* ones.

EQUALITY OF GENERATING FUNCTIONS

Two generating functions $f(x)=\sum_{n=0}^{\infty}a_nx^n$ and $g(x)=\sum_{n=0}^{\infty}b_nx^n$ are equal if $a_n=b_n$ for $n\geq 0$; that is $\sum_{n=0}^{\infty}a_nx^n=\sum_{n=0}^{\infty}b_nx^n \Rightarrow a_n=b_n$ for $n\geq 0$.

ADDITION AND MULTIPLICATION OF GENERATING FUNCTIONS

Let $f(x)=\sum_{n=0}^{\infty}a_nx^n$ and $g(x)=\sum_{n=0}^{\infty}b_nx^n$ be two generating functions. Then

$$f(x)+g(x)=\sum_{n=0}^{\infty}(a_n+b_n)x^n$$

And
$$f(x)g(x)=\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}a_ib_{n-i}\right)x^n \qquad (2)$$

GENERATING FUNCTION FOR P_n:

First, we develop a generating function $g(x)$ for P_n. So, let $g(x)=\sum_{n=0}^{\infty}P_nx^n$;

$$g(x)=P_0+P_1x+P_2x^2+....+P_nx^n+P_{n+1}x^{n+1}+....$$
$$2xg(x)=2P_0x+2P_1x^2+....+2P_{n-1}x^n+2P_nx^{n+1}+....$$
$$x^2g(x)=P_0x^2+....+P_{n-2}x^n+P_{n-1}x^{n+1}+....$$
$$\therefore\ (1-2x-x^2)g(x)=P_0+P_1x-2P_0x=2x \qquad (\text{Since } P_{n+1}=2P_n+P_{n-1})$$

Using $P_0=0\ and\ P_1=1,$ we get

$$\sum_{n=0}^{\infty}P_nx^n=\frac{x}{(1-2x-x^2)}. \qquad (3)$$

Therefore, $\dfrac{x}{(1-2x-x^2)}$ generates the sequence of Pell numbers.

GENERATING FUNCTION FOR $P_{n+1}P_{n+2}$:

Let $g(x)=\sum_{n=0}^{\infty}P_{n+1}P_{n+2}x^n$; so that

$$g(x)=P_1P_2+P_2P_3x+P_3P_4x^2+P_4P_5x^3+....+P_{n+1}P_{n+2}x^n+P_{n+2}P_{n+3}x^{n+1}+....$$
$$5xg(x)=5P_1P_2x+5P_2P_3x^2+5P_3P_4x^3+....+5P_nP_{n+1}x^n+5P_{n+1}P_{n+2}x^{n+1}+....$$
$$5x^2g(x)=5P_1P_2x^2+5P_2P_3x^3+....+5P_{n-1}P_nx^n+5P_nP_{n+1}x^{n+1}+....$$
$$x^3g(x)=P_1P_2x^3+....+P_{n-2}P_{n-1}x^n+P_{n-1}P_nx^{n+1}+....$$
$$\therefore\ (1-5x-5x^2+x^3)g(x)=\left(P_1P_2+P_2P_3x+P_3P_4x^2\right)-\left(5P_1P_2x+5P_2P_3x^2\right)-\left(5P_1P_2x^2\right)$$
$$\left[\because P_{n+2}P_{n+3}+P_{n-1}P_n=5\left(P_{n+1}P_{n+2}+P_nP_{n+1}\right)\right]$$
$$=\left(2+10x+60x^2\right)-\left(10x+50x^2\right)-\left(10x^2\right)$$

Thus, $\sum_{n=0}^{\infty} P_{n+1}P_{n+2}x^n = \dfrac{2}{(1-5x-5x^2+x^3)}$.

GENERATING FUNCTION FOR P_n^2 :

Let $g(x)=\sum_{n=0}^{\infty} P_n^2 x^n$; so that

$$g(x)=P_0^2+P_1^2x+P_2^2x^2+P_3^2x^3+P_4^2x^4+....+P_n^2x^n+P_{n+1}^2x^{n+1}+....$$

$$5xg(x)=5P_0^2x+5P_1^2x^2+5P_2^2x^3+5P_3^2x^4+....+5P_{n-1}^2x^n+5P_n^2x^{n+1}+....$$

$$5x^2g(x)=5P_0^2x^2+5P_1^2x^3+5P_2^2x^4+....+5P_{n-2}^2x^n+5P_{n-1}^2x^{n+1}+....$$

$$x^3g(x)=P_0^2x^3+P_1^2x^4+....+P_{n-3}^2x^n+P_{n-2}^2x^{n+1}+....$$

$$\therefore\ (1-5x-5x^2+x^3)g(x)=\left(P_0^2+P_1^2x+P_2^2x^2\right)-\left(5P_0^2x+5P_1^2x^2\right)-\left(5P_0^2x^2\right)$$

$$\left[\because P_{n+1}^2+P_{n+2}^2=5\left(P_n^2+P_{n+1}^2\right)\right]$$

$$=\left(0+x+4x^2\right)-\left(0x+5x^2\right)-\left(0x^2\right)=x\left(1-x\right)$$

Thus, $\sum_{n=0}^{\infty} P_n^2x^n=\dfrac{x\left(1-x\right)}{(1-5x-5x^2+x^3)}$.

GENERATING FUNCTION FOR P_n^3 :

Let $g(x)=\sum_{n=0}^{\infty} P_n^3 x^n$; so that

$$g(x)=P_0^3+P_1^3x+P_2^3x^2+P_3^3x^3+P_4^3x^4+P_5^3x^5+....+P_n^3x^n+P_{n+1}^3x^{n+1}+....$$

$$12xg(x)=12P_0^3x+12P_1^3x^2+12P_2^3x^3+12P_3^3x^4+12P_4^3x^5+....+12P_{n-1}^3x^n+12P_n^3x^{n+1}+....$$

$$30x^2g(x)=30P_0^3x^2+30P_1^3x^3+30P_2^3x^4+30P_3^3x^5+....+30P_{n-2}^3x^n+30P_{n-1}^3x^{n+1}+....$$

$$12x^3g(x)=12P_0^3x^3+12P_1^3x^4+12P_2^3x^5+....+12\ P_{n-3}^3x^n+12P_{n-2}^3x^{n+1}+....$$

$$x^4g(x)=P_0^3x^4+P_1^3x^5+....+P_{n-4}^3x^n+P_{n-3}^3x^{n+1}+....$$

$$\therefore\ (1-12x-30x^2+12x^3+x^4)g(x)=\left(P_0^3+P_1^3x+P_2^3x^2+P_3^3x^3\right)-\left(12P_0^3x+12P_1^3x^2+12P_2^3x^2\right)$$

$$-\left(30P_0^3x^2+30P_1^3x^3\right)+\left(12P_0^3x^3\right)$$

$$\left[\because P_{n+4}^3-12P_{n+3}^3-30P_{n+2}^3+12P_{n+1}^3+P_n^3=0\right]$$

$$=\left(0+x+8x^2+125x^3\right)-\left(0x+12x^2+96x^3\right)-\left(0x^2+30x^3\right)+\left(0x^3\right)$$

$$=x-4x^2-x^3$$

Thus, $\sum_{n=0}^{\infty} P_n^3x^n=\dfrac{x\left(1-4x-x^2\right)}{(1-12x-30x^2+12x^3+x^4)}$.

A LIST OF GENERATING FUNCTIONS

Using the above technique and Pell Identities, we can also derive the following generating functions:

(i) $\dfrac{1-2x}{(1-2x-x^2)}=\sum_{n=0}^{\infty} P_{n-1}x^n$ (4)

(ii) $\dfrac{1}{(1-2x-x^2)}=\sum_{n=0}^{\infty} P_{n+1}x^n$ (5)

(iii) $$\frac{2(1-x)}{(1-2x-x^2)}=\sum_{n=0}^{\infty}Q_n x^n \qquad (6)$$

(iv) $$\frac{2(1+x)}{(1-2x-x^2)}=\sum_{n=0}^{\infty}Q_{n+1} x^n \qquad (7)$$

(v) $$\frac{-2+6x}{(1-2x-x^2)}=\sum_{n=0}^{\infty}Q_{n-1} x^n$$

(vi) $$\frac{2x}{(1-6x+x^2)}=\sum_{n=0}^{\infty}P_{2n} x^n \qquad \left(\text{Using } P_{2n+2}+P_{2n-2}=6P_{2n},\, n\geq 1\right)$$

(vii) $$\frac{1-x}{(1-6x+x^2)}=\sum_{n=0}^{\infty}P_{2n+1} x^n \qquad \left(\text{Using } P_{2n+3}+P_{2n-1}=6P_{2n+1},\, n\geq 1\right)$$

(viii) $$\frac{2(1-3x)}{(1-6x+x^2)}=\sum_{n=0}^{\infty}Q_{2n} x^n \qquad \left(\text{Using } Q_{2n+2}+Q_{2n-2}=6Q_{2n},\, n\geq 1\right)$$

(ix) $$\frac{2(1+x)}{(1-6x+x^2)}=\sum_{n=0}^{\infty}Q_{2n+1} x^n \qquad \left(\text{Using } Q_{2n+3}+Q_{2n-1}=6Q_{2n+1},\, n\geq 1\right)$$

(x) $$\frac{2(3-x)}{(1-6x+x^2)}=\sum_{n=0}^{\infty}Q_{2n+2} x^n$$

(xi) $$\frac{2}{(1-6x+x^2)}=\sum_{n=0}^{\infty}P_{2n+2} x^n$$

(xii) $$\frac{5x}{(1-14x-x^2)}=\sum_{n=0}^{\infty}P_{3n} x^n \qquad \left(\text{Using } P_{3n+3}-P_{3n-3}=14P_{3n},\, n\geq 1\right)$$

(xiii) $$\frac{2-14x}{(1-14x-x^2)}=\sum_{n=0}^{\infty}Q_{3n} x^n \qquad \left(\text{Using } Q_{3n+3}-Q_{3n-3}=14Q_{3n},\, n\geq 1\right)$$

(xiv) $$\frac{1-x}{(1-5x-5x^2+x^3)}=\sum_{n=0}^{\infty}P_{n+1}^2 x^n$$

(xv) $$\frac{4+5x-x^2}{(1-5x-5x^2+x^3)}=\sum_{n=0}^{\infty}P_{n+2}^2 x^n$$

(xvi) $$\frac{4-16x-4x^2}{(1-5x-5x^2+x^3)}=\sum_{n=0}^{\infty}Q_n^2 x^n$$

(xvii) $$\frac{4+16x-4x^2}{(1-5x-5x^2+x^3)}=\sum_{n=0}^{\infty}Q_{n+1}^2 x^n$$

(xviii) $$\frac{36+16x-4x^2}{(1-5x-5x^2+x^3)}=\sum_{n=0}^{\infty}Q_{n+2}^2 x^n$$

(xix) $$\frac{8-88x-120x^2+8x^3}{(1-12x-30x^2+12x^3+x^4)}=\sum_{n=0}^{\infty}Q_n^3 x^n$$

(xx) $$\frac{1-4x-x^2}{(1-12x-30x^2+12x^3+x^4)}=\sum_{n=0}^{\infty}P_{n+1}^3 x^n$$

(xxi) $$\frac{8+120x-88x^2-8x^3}{(1-12x-30x^2+12x^3+x^4)}=\sum_{n=0}^{\infty}Q_{n+1}^3 x^n$$

(xxii) $$\frac{8+29x-12x^2-x^3}{(1-12x-30x^2+12x^3+x^4)}=\sum_{n=0}^{\infty}P_{n+2}^3x^n$$

(xxiii) $$\frac{216+152x-104x^2-8x^3}{(1-12x-30x^2+12x^3+x^4)}=\sum_{n=0}^{\infty}Q_{n+2}^3x^n$$

(xxiv) $$\frac{10x}{(1-12x-30x^2+12x^3+x^4)}=\sum_{n=0}^{\infty}P_n P_{n+1}P_{n+2}x^n$$

(xxv) $$\frac{24-120x+120x^2+8x^3}{(1-12x-30x^2+12x^3+x^4)}=\sum_{n=0}^{\infty}Q_nQ_{n+1}Q_{n+2}x^n$$

(xxvi) $$\frac{P_k x}{1-Q_k x+(-1)^k x^2}=\sum_{n=0}^{\infty}P_{nk}x^n$$

(xxvii) $$\frac{P_r+(-1)^r P_{k-r}x}{1-Q_k x+(-1)^k x^2}=\sum_{n=0}^{\infty}P_{nk+r}x^n$$

(xxviii) $$\frac{2+3Q_k x}{1-Q_k x+(-1)^k x^2}=\sum_{n=0}^{\infty}Q_{nk}x^n$$

(xxvii) $$\frac{Q_r+(-1)^r Q_{k-r}x}{1-Q_k x+(-1)^k x^2}=\sum_{n=0}^{\infty}Q_{nk+r}x^n$$

GENERATING FUNCTIONS FOR s_n:

Let $s_n=\sum_{i=1}^{n}P_i$ and $g(x)=\sum_{n=0}^{\infty}s_nx^n$; then

$$\sum_{n=0}^{\infty}s_nx^n=\sum_{n=0}^{\infty}\frac{1}{2}\left(P_n+P_{n+1}-1\right)x^n$$ [Since $\sum_{i=1}^{n}P_i=\frac{1}{2}\left(P_n+P_{n+1}-1\right)$]

$$=\frac{1}{2}\left[\sum_{n=0}^{\infty}P_nx^n+\sum_{n=0}^{\infty}P_{n+1}x^n-\sum_{n=0}^{\infty}x^n\right]$$

$$=\frac{1}{2}\left[\frac{x}{(1-2x-x^2)}+\frac{1}{(1-2x-x^2)}-\frac{1}{1-x}\right]$$ [Using (1), (3) and (5)]

$$=\frac{1}{2}\left[\frac{2x}{(1-3x+x^2+x^3)}\right]$$

`Thus, $$\sum_{n=0}^{\infty}s_nx^n=\frac{x}{(1-3x+x^2+x^3)}.$$

GENERATING FUNCTIONS FOR P_{m+n} and Q_{m+n}:

Using Binet formula, generating function for P_{m+n} can be derived as follows:

$$\sum_{n=0}^{\infty}P_{m+n}x^n=\sum_{n=0}^{\infty}\left(\frac{\gamma^{m+n}-\delta^{m+n}}{\gamma-\delta}\right)x^n$$

$$=\frac{1}{\gamma-\delta}\left[\gamma^m\sum_{n=0}^{\infty}\gamma^nx^n-\delta^m\sum_{n=0}^{\infty}\delta^nx^n\right]$$

$$=\frac{1}{\gamma-\delta}\left[\frac{\gamma^m}{1-\gamma x}-\frac{\delta^m}{1-\delta x}\right]$$ [Since $\sum_{n=0}^{\infty}a^nx^n=\frac{1}{1-ax}$]

$$=\frac{1}{\gamma-\delta}\left[\frac{\gamma^m(1-\delta x)-\delta^m(1-\gamma x)}{(1-\gamma x)(1-\delta x)}\right]$$

$$=\frac{1}{\gamma-\delta}\left[\frac{(\gamma^m-\delta^m)-\gamma\delta x(\gamma^{m-1}-\delta^{m-1})}{1-(\gamma+\delta)x+\gamma\delta x^2}\right]$$

$$=\frac{1}{\gamma-\delta}\left[\frac{(\gamma^m-\delta^m)+(\gamma^{m-1}-\delta^{m-1})x}{1-(\gamma+\delta)x+\gamma\delta x^2}\right]$$

$$=\frac{P_m+P_{m-1}x}{1-2x-x^2}$$

Thus, $\sum_{n=0}^{\infty}P_{m+n}x^n=\left(\frac{P_m+P_{m-1}x}{1-2x-x^2}\right)$.

Likewise, it can be shown that $\sum_{n=0}^{\infty}Q_{m+n}x^n=\left(\frac{Q_m+Q_{m-1}x}{1-2x-x^2}\right)$.

EXPONENTIAL GENERATING FUNCTIONS

Since $e^t=\sum_{n=0}^{\infty}\frac{t^n}{n!}$, it follows that $e^{\gamma x}=\sum_{n=0}^{\infty}\frac{\gamma^n x^n}{n!}$ and $e^{\delta x}=\sum_{n=0}^{\infty}\frac{\delta^n x^n}{n!}$

$$\therefore\ \frac{e^{\gamma x}-e^{\delta x}}{\gamma-\delta}=\sum_{n=0}^{\infty}\left(\frac{\gamma^n-\delta^n}{\gamma-\delta}\right)\frac{x^n}{n!}=\sum_{n=0}^{\infty}P_n\frac{x^n}{n!}. \qquad (8)$$

Thus, the exponential function $\frac{e^{\gamma x}-e^{\delta x}}{\gamma-\delta}$ generates the numbers $\frac{P_n}{n!}$.

Likewise, we can show that

$$e^{\gamma x}+e^{\delta x}=\sum_{n=0}^{\infty}Q_n\frac{x^n}{n!}; \qquad (9)$$

That is $(e^{\gamma x}+e^{\delta x})$ generates the numbers $\frac{Q_n}{n!}$.

More generally, we can show that

$$\frac{e^{\gamma^k x}-e^{\delta^k x}}{\gamma-\delta}=\sum_{n=0}^{\infty}\frac{P_{nk}}{n!}x^n. \qquad (10)$$

Likewise,

$$e^{\gamma^k x}+e^{\delta^k x}=\sum_{n=0}^{\infty}\frac{Q_{nk}}{n!}x^n. \qquad (11)$$

Using (8), we can write

$$\frac{e^{(1+\sqrt{2})x}-e^{(1-\sqrt{2})x}}{(1+\sqrt{2})-(1-\sqrt{2})}=\sum_{n=0}^{\infty}P_n\frac{x^n}{n!}$$

Or

$$e^{(1+\sqrt{2})x}-e^{(1-\sqrt{2})x}=2\sqrt{2}\sum_{n=0}^{\infty}P_n\frac{x^n}{n!}$$

Or

$$e^x\left[e^{\sqrt{2}x}-e^{-\sqrt{2}x}\right]=2\sqrt{2}\sum_{n=0}^{\infty}P_n\frac{x^n}{n!}$$

Or

$$2e^x\sinh\sqrt{2}x=2\sqrt{2}\sum_{n=0}^{\infty}P_n\frac{x^n}{n!}$$

Or

Thus,

$$e^x \sinh \sqrt{2}x = \sqrt{2}\sum_{n=0}^{\infty} P_n \frac{x^n}{n!}$$

Similarly,

$$2e^x \cosh \sqrt{2}x = \sum_{n=0}^{\infty} Q_n \frac{x^n}{n!}.$$

IDENTITIES USING GENERATING FUNCTIONS

We have $\sum_{n=0}^{\infty} P_{n+1}x^n = \frac{1}{D}$, $\sum_{n=0}^{\infty} P_{n-1}x^n = \frac{1-2x}{D}$ and $\sum_{n=0}^{\infty} Q_n x^n = \frac{2(1-x)}{D}$ where $D = 1-2x-x^2$. Also, $\frac{2(1-x)}{D} = \frac{1}{D} + \frac{1-2x}{D}$.

It follows that $\sum_{n=0}^{\infty} \frac{2(1-x)}{D} = \sum_{n=0}^{\infty} \frac{1}{D} + \sum_{n=0}^{\infty} \frac{1-2x}{D}$

$$\Rightarrow \sum_{n=0}^{\infty} Q_n x^n = \sum_{n=0}^{\infty} P_{n+1}x^n + \sum_{n=0}^{\infty} P_{n-1}x^n$$

$$= \sum_{n=0}^{\infty} (P_{n+1} + P_{n-1})x^n$$

$$\therefore \quad Q_n = P_{n+1} + P_{n-1}.$$

Next we shall prove that $P_m Q_n + P_{m-1}Q_{n-1} = Q_{m+n-1}$.

$$\sum_{m=0}^{\infty} (P_m Q_n + P_{m-1}Q_{n-1})x^m = Q_n \sum_{m=0}^{\infty} P_m x^m + Q_{n-1}\sum_{m=0}^{\infty} P_{m-1}x^m$$

$$= Q_n \frac{x}{D} + Q_{n-1}\frac{1-2x}{D}$$

$$= \frac{Q_{n-1} + (Q_n - 2Q_{n-1})x}{D} = \frac{Q_{n-1} + Q_{n-2}x}{D}$$

$$= \sum_{m=0}^{\infty} Q_{m+n-1}x^m$$

$$\therefore \quad P_m Q_n + P_{m-1}Q_{n-1} = Q_{m+n-1}.$$

Similarly, we can prove other identities like

$P_m P_n + P_{m-1}P_{n-1} = P_{m+n-1}$, $Q_m Q_n + Q_{m-1}Q_{n-1} = 8P_{m+n-1}$ etc.

COMBINATORIAL IDENTITIES USING GENERATING FUNCTIONS

Let $A(t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!}$ and $B(t) = \sum_{n=0}^{\infty} b_n \frac{t^n}{n!}$; then

$$A(t)B(t) = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}a_k b_{n-k}\right]\frac{t^n}{n!}, \quad (12)$$

$$A(t)B(-t) = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}a_k b_{n-k}\right]\frac{t^n}{n!} \quad (13)$$

and

$$A(2t)B(t) = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}2^k\binom{n}{k}a_k b_{n-k}\right]\frac{t^n}{n!}. \quad (14)$$

In particular, $A(t)=\dfrac{e^{\gamma t}-e^{\delta t}}{\gamma-\delta}$ and $B(t)=e^{t}$; then

$$A(2t)B(t)=\left(\frac{e^{2\gamma t}-e^{2\delta t}}{\gamma-\delta}\right)e^{t}$$

$$\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}2^{k}\binom{n}{k}a_k b_{n-k}\right]\frac{t^n}{n!}=\left(\frac{e^{2\gamma t}-e^{2\delta t}}{\gamma-\delta}\right)e^{t}$$ By (14)

$$\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}2^{k}\binom{n}{k}P_k\right]\frac{t^n}{n!}=\left(\frac{e^{(1+2\gamma)t}-e^{(1+2\delta)t}}{\gamma-\delta}\right)$$ By (8)

$$=\left(\frac{e^{\gamma^2 t}-e^{\delta^2 t}}{\gamma-\delta}\right)$$ (Since $\gamma^2-2\gamma-1=0$)

$$=\sum_{n=0}^{\infty}P_{2n}\frac{t^n}{n!}$$ By (10)

Thus,

$$\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}2^{k}\binom{n}{k}P_k\right]\frac{t^n}{n!}=\sum_{n=0}^{\infty}P_{2n}\frac{t^n}{n!}$$

Equating the coefficients of $\dfrac{t^n}{n!}$ yields the combinatorial identity

$$\sum_{k=0}^{n}2^{k}\binom{n}{k}P_k=P_{2n}$$

Now taking, $A(t)=\dfrac{e^{\gamma t}-e^{\delta t}}{\gamma-\delta}$ and $B(t)=e^{-t}$;

$$A(t)B(2t)=\left(\frac{e^{\gamma t}-e^{\delta t}}{\gamma-\delta}\right)e^{-2t}$$

$$\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}(-2)^{n-k}\binom{n}{k}a_k b_{n-k}\right]\frac{t^n}{n!}=\left(\frac{e^{\gamma t}-e^{\delta t}}{\gamma-\delta}\right)e^{-2t}$$

$$\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}(-2)^{n-k}\binom{n}{k}P_k\right]\frac{t^n}{n!}=\left(\frac{e^{(-2+\gamma)t}-e^{(-2+\delta)t}}{\gamma-\delta}\right)$$

$$=\left(\frac{e^{-\delta t}-e^{-\gamma t}}{\gamma-\delta}\right)$$ (Since $\gamma+\delta=2$)

$$=\sum_{n=0}^{\infty}(-1)P_n\frac{(-t)^n}{n!}=\sum_{n=0}^{\infty}(-1)^{n-1}P_n\frac{t^n}{n!}$$

Thus,

$$\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}(-2)^{n-k}\binom{n}{k}P_k\right]\frac{t^n}{n!}=\sum_{n=0}^{\infty}(-1)^{n-1}P_n\frac{t^n}{n!}$$

Equating the coefficients of $\dfrac{t^n}{n!}$ yields another combinatorial identity

$$\sum_{k=0}^{n}(-2)^{n-k}\binom{n}{k}P_k=(-1)^{n-1}P_n.$$

Obviously, by selecting $A(t)$ and $B(t)$ as suitable exponential functions, we can apply this method to derive other Pell and Pell-Lucas combinatorial identities.

For example, choosing $A(t)=e^{\gamma t}+e^{\delta t}$ and $B(t)=e^{t}$, we get $\sum_{k=0}^{n} 2^k \binom{n}{k} Q_k = Q_{2n}$; and taking $A(t)=e^{\gamma t}+e^{\delta t}$ and $B(t)=e^{-t}$, we get $\sum_{k=0}^{n}(-2)^{n-k}\binom{n}{k}Q_k=(-1)^n Q_n$.

COMBINATORIAL IDENTITIES USING THE DIFFERENTIAL OPERATOR

Since $A(t)=\sum_{n=0}^{\infty} a_n \frac{t^n}{n!}$, it follows that $\frac{d^r}{dt^r}A(t)=\sum_{n=0}^{\infty} a_{n+r}\frac{t^n}{n!}$ (15)

In particular, taking $A(t)=\frac{e^{2\alpha t}-e^{2\beta t}}{\alpha-\beta}$ and $B(t)=e^{t}$; we get

$$A(t)B(t)=\left(\frac{e^{2\alpha t}-e^{2\beta t}}{\alpha-\beta}\right)e^{t}$$

$$A(t)B(t)=\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}2^k\binom{n}{k}P_k\right]\frac{t^n}{n!}$$

$$\left(\because \quad A(t)B(t)=\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}a_k b_{n-k}\right]\frac{t^n}{n!}\right)$$

Using (15) we can write

$$B(t)\frac{d^r}{dt^t}A(t)=\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}2^{k+r}\binom{n}{k}P_{k+r}\right]\frac{t^n}{n!}$$

Or

$$\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}2^{k+r}\binom{n}{k}P_{k+r}\right]\frac{t^n}{n!}=B(t)\frac{d^r}{dt^t}\left(\frac{e^{2\alpha t}-e^{2\beta t}}{\alpha-\beta}\right)$$

$$=e^{t}.\left[\frac{(2\alpha)^r e^{2\alpha t}-(2\beta)^r e^{2\beta t}}{\alpha-\beta}\right]$$

$$=\frac{(2\alpha)^r e^{(2\alpha+1)t}-(2\beta)^r e^{(2\beta+1)t}}{\alpha-\beta}$$

$$=2^r\left(\frac{\alpha^r e^{\alpha^2 t}-\beta^r e^{\beta^2 t}}{\alpha-\beta}\right)$$ (Since $\alpha^2-2\alpha-1=0$)

$$=2^r\sum_{n=0}^{\infty}P_{2n+r}\frac{t^n}{n!}$$

Thus,

$$\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}2^{k+r}\binom{n}{k}P_{k+r}\right]\frac{t^n}{n!}=2^r\sum_{n=0}^{\infty}P_{2n+r}\frac{t^n}{n!}.$$

Equating the coefficients of $\frac{t^n}{n!}$ yields the combinatorial identity

$$\sum_{k=0}^{n}2^{k+r}\binom{n}{k}P_{k+r}=2^r P_{2n+r}.$$

Or

$$\sum_{k=0}^{n}2^{k}\binom{n}{k}P_{k+r}=P_{2n+r}.$$

HYBRID IDENTITIES

Here we derive some hybrid identities involving both Pell and Pell-Lucas numbers. If we choose $A(t)=\dfrac{e^{\gamma t}-e^{\delta t}}{\gamma-\delta}$ and $B(t)=e^{\gamma t}+e^{\delta t}$; then

$$A(t)B(t)=\left(\frac{e^{\gamma t}-e^{\delta t}}{\gamma-\delta}\right)\left(e^{\gamma t}+e^{\delta t}\right)$$

Or
$$\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}P_kQ_{n-k}\right]\frac{t^n}{n!}=\left(\frac{e^{2\gamma t}-e^{2\delta t}}{\gamma-\delta}\right)$$
[By (8) and (12)]

Or
$$\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}P_kQ_{n-k}\right]\frac{t^n}{n!}=\sum_{n=0}^{\infty}P_n\frac{(2t)^n}{n!}$$

Or
$$\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}P_kQ_{n-k}\right]\frac{t^n}{n!}=\sum_{n=0}^{\infty}2^nP_n\frac{t^n}{n!}$$

This yields the combinatorial hybrid identity that

$$\sum_{k=0}^{n}\binom{n}{k}P_kQ_{n-k}=2^nP_n \qquad (16)$$

By choosing $A(t)=\dfrac{e^{\gamma t}-e^{\delta t}}{\gamma-\delta}=B(t)$, we get

$$A(t)B(t)=\left(\frac{e^{\gamma t}-e^{\delta t}}{\gamma-\delta}\right)\left(\frac{e^{\gamma t}-e^{\delta t}}{\gamma-\delta}\right)$$

Or
$$\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}P_kP_{n-k}\right]\frac{t^n}{n!}=\left(\frac{e^{2\gamma t}+e^{2\delta t}-2e^{(\gamma+\delta)t}}{(\gamma-\delta)^2}\right)$$
$$=\frac{1}{8}\left[\left(e^{2\gamma t}+e^{2\delta t}\right)-2e^{2t}\right]$$
$$=\sum_{n=0}^{\infty}\frac{1}{8}\left[2^nQ_n-2.2^n\right]\frac{t^n}{n!}$$
$$=\sum_{n=0}^{\infty}2^{n-3}\left(Q_n-2\right)\frac{t^n}{n!}$$

Thus,
$$\sum_{k=0}^{n}\binom{n}{k}P_kP_{n-k}=2^{n-3}\left(Q_n-2\right) \qquad (17)$$

Similarly,
$$\sum_{k=0}^{n}\binom{n}{k}Q_kQ_{n-k}=2^n\left(Q_n+2\right) \qquad (18)$$

GENERALIZED HYBRID IDENTITIES

Identities (16) through (18) can also be generalized as follows:

$$\sum_{k=0}^{n}\binom{n}{k}P_{mk}Q_{mn-mk}=2^nP_{mn}$$
$$\sum_{k=0}^{n}\binom{n}{k}P_{mk}P_{mn-mk}=\frac{2^nQ_{mn}-2Q_m^n}{8}$$
$$\sum_{k=0}^{n}\binom{n}{k}Q_{mk}Q_{mn-mk}=2^nQ_{mn}+2Q_m^n$$

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